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Appendix A: Techniques for Quadcopter Modelling & Design: A Review

OUR QUADCOPTER EQUATION LINEARIZATION

TAYLOR SERIES APPROXIMATION

First order Taylor series approximation was used to linearize the quadcopter's non-linear dynamic equations-of-motions to a linear model for use with linear controllers like the PID-controller. The approximation was done by representing the given equation as a function with a series of terms calculated from the function's derivatives at a single point known as the operating point.

$$f(a) \approx f(\bar{a}) + \frac{f'(\bar{a})}{1!}(a - \bar{a})^1 + \frac{f''(\bar{a})}{2!}(a - \bar{a})^2 + \frac{f'''(\bar{a})}{3!}(a - \bar{a})^3 + \dots + \frac{f^n(\bar{a})}{n!}(a - \bar{a})^n$$

In linearizing the system equation-of-motions, the first order Taylor approximation used is written as:

$$f(a) \approx f(\bar{a}) + f'(a - \bar{a})|_{n=1}$$

In the case of the quad-rotor the normal behaviour is when it is in stable hover. This implies that the vehicle's x-y body plane is parallel to the x-y earth plane, equivalent to a situation when the vehicle's roll, pitch and yaw angles are all equal to or close to zero. The operating input voltage that corresponds to achieving the above conditions can be calculated from the quad-rotor system equations. Here, all operating points are represented with subscript 'h'.

$$\ddot{z} = \frac{2\rho A}{M} \left[\frac{f\eta K_t}{K_q} \right]^2 (V_1^2 + V_2^2 + V_3^2 + V_4^2) (\cos\theta \cos\phi) - g$$

With

$$\left\{ \begin{array}{l} \theta = \phi = \psi = \dot{\theta} = \dot{\phi} = \dot{\psi} = \ddot{\theta} = \ddot{\phi} = \ddot{\psi} = 0 \\ \Omega = \Omega_h \\ \dot{\Omega} = \ddot{\Omega} = 0 \\ x = \dot{x} = \ddot{x} = y = \dot{y} = \ddot{y} = z = \dot{z} = \ddot{z} = 0 \end{array} \right.$$

Substituting the constants values from table-1 and the conditions listed above, the resulting equation becomes:

$$0 = \left(\frac{2.2 \cdot 0.08042}{0.65} \right) \left(\frac{0.005 \cdot 0.75}{0.0056} \right)^2 (V_1^2 + V_2^2 + V_3^2 + V_4^2) (\cos\theta \cos\phi) - 9.81$$

$$0 = 0.27219(0.44842)(V_1^2 + V_2^2 + V_3^2 + V_4^2) - 9.81$$

$$0 = 0.12206(V_1^2 + V_2^2 + V_3^2 + V_4^2) - 9.81$$

$$V_h = V_1 = V_2 = V_3 = V_4$$

$$0 = 0.12206(4V_h)^2 - 9.81$$

$$V_h = \sqrt{20.0933}$$

$$V_h = 4.48 \text{ volts}$$

For this voltage, the speed with which the motors spin can be found at the operating point by substituting using the voltage equation for the rotor, which describes the relationship between the motor speed and voltage input.

$$V = J\Omega \frac{R}{K_e} + K_e\Omega + K_q R \Omega^2$$

At the operating point voltage, let the motor speed be Ω_h

$$V_h = \left(\frac{0.00006 * 0.6}{0.0056} \right) \frac{d\Omega}{dt} + 0.0015\Omega + \left(\frac{0.00006 * 0.6}{0.0056} \right) \Omega^2$$

$$4.48 = 0.00643\Omega + 0.0015\Omega + 0.0000804\Omega^2$$

$$\frac{d\Omega}{dt} = 696.73406 - 0.23328\Omega - 0.0000804\Omega^2$$

Since the motor speeds remain constant at hover, $\frac{d\Omega}{dt} = 0$ and on substituting, we obtained the following quadratic equation:

$$696.73406 - 0.23328\Omega - 0.0000804\Omega^2 = 0$$

$$\Omega_h = \frac{-(-0.23328) \pm \sqrt{(-0.23328)^2 - [4 * (-0.01250) * 696.73406]}}{2 * (-0.01250)}$$

$$\Omega_h = \frac{(0.23328) \pm \sqrt{34.89112}}{-0.025}$$

$$\Omega_h = -245.61 \text{ rads}^{-1} \text{ or } 226.94 \text{ rads}^{-1}$$

By discarding the negative root of the quadratic equation, the speed at operating voltage is

$$\Omega_h = 226.94 \text{ rads}^{-1}$$

Linearization of Quad-rotor Equations

From our voltage equation, $V_h = V = 0.00643 \frac{d\Omega}{dt} + 0.0015 \Omega + 0.0000804 \Omega^2$

Let $A = 0.00643$, $B = 0.0015$ and $C = 0.0000804$

This leads to: $V = A \frac{d\Omega}{dt} + B \Omega + C \Omega^2$

Making $\frac{d\Omega}{dt}$ the subject of the formula, gives:

$$\frac{d\Omega}{dx} = \frac{V}{A} - \frac{B}{A} \Omega - \frac{C}{A} \Omega^2 = f(\Omega, V)$$

First, the coefficients of the first order Taylor series have to be found.

$$\dot{L} = \left. \frac{\partial f}{\partial \Omega} \right|_h = -\frac{B}{A} - \frac{2C}{A} \Omega_h = \frac{-B-2C}{A} \Omega_h$$

$$\dot{M} = \left. \frac{\partial f}{\partial v} \right|_h = \frac{1}{A}$$

$$\frac{d(\Delta\Omega)}{dt} = \dot{L}\Delta\Omega + \dot{M}\Delta V$$

$$\frac{d(\Delta\Omega)}{dt} = \frac{-B-2C}{A} \Omega_h \Delta\Omega + \frac{1}{A} \Delta V$$

$$\frac{d(\Delta\Omega)}{dt} = -5.909\Delta\Omega + 155.521\Delta V = 155.521\Delta V - 5.909 \Delta\Omega$$

Where $\Delta\Omega$ is $\Omega(t) - \Omega_h$ and $\Delta V = V(t) - V_h$.

For the roll angular acceleration,

$$\frac{d^2\varphi}{dt^2} = 4.52245(V_2^2 - V_4^2). \text{ Where } A = 4.52245 \text{ then,}$$

$$\frac{d^2\varphi}{dt^2} = A(V_2^2 - V_4^2) = f(V_2, V_4)$$

At the operating point, $\varphi = \frac{d\varphi}{dt} = \frac{d^2\varphi}{dt^2} = 0$

$$\dot{L} = \left. \frac{\partial f}{\partial \varphi} \right|_h = 0$$

$$\dot{M} = \left. \frac{\partial f}{\partial V_2} \right|_h = 2AV_2 = 2AV_h$$

$$\dot{N} = \left. \frac{\partial f}{\partial V_4} \right|_h = -2AV_4 = -2AV_h$$

$$\frac{d\varphi}{dt} \left(\frac{d(\Delta\varphi)}{dt} \right) = \dot{L}\Delta\varphi + \dot{M}\Delta V_2 + \dot{N}\Delta V_4$$

$$\frac{d\varphi}{dt} \left(\frac{d(\Delta\varphi)}{dt} \right) = 0 + 2AV_h\Delta V_2 - 2AV_h\Delta V_4 = 2AV_h(\Delta V_2 - \Delta V_4)$$

$$\frac{d\varphi}{dt} \left(\frac{d(\Delta\varphi)}{dt} \right) = 2052.6496(\Delta V_2 - \Delta V_4)$$

Similarly, for the pitch angular acceleration,

$$\frac{d^2\theta}{dt^2} = 4.52245(V_3^2 - V_1^2)$$

This result in

$$\frac{d\theta}{dt} \left(\frac{d(\Delta\theta)}{dt} \right) = 0 + 2AV_h\Delta V_3 - 2AV_h\Delta V_1 = 2AV_h(\Delta V_3 - \Delta V_1)$$

$$\frac{d\theta}{dt} \left(\frac{d(\Delta\theta)}{dt} \right) = 2052.6496(\Delta V_3 - \Delta V_1)$$

For yaw angular acceleration,

$$\frac{d\psi}{dt} \left(\frac{d(\Delta\psi)}{dt} \right) = 0.004615 \left[\frac{d}{dt}\Omega_1 - \frac{d}{dt}\Omega_2 + \frac{d}{dt}\Omega_3 - \frac{d}{dt}\Omega_4 \right] + 0.0000577[\Omega_1^2 - \Omega_2^2 + \Omega_3^2 - \Omega_4^2]$$

Let A = 0.004615 and B = 0.0000577. At the operating point, $\psi = \frac{d\psi}{dt} = \frac{d\psi}{dt} \left(\frac{d(\Delta\psi)}{dt} \right) = 0$, since all the

$$\dot{L} = \frac{\partial f}{\partial \psi} \Big|_h = 0; \quad \dot{M} = \frac{\partial f}{\partial \Omega} \Big|_h = 2B\Omega_1 = 2B\Omega_h; \quad \dot{N} = \frac{\partial f}{\partial \Omega_2} \Big|_h = -2B\Omega_2 =$$

$$2B\Omega_h;$$

$$\dot{O} = \frac{\partial f}{\partial \Omega_3} \Big|_h = -2B\Omega_3 = 2B\Omega_h; \quad \dot{P} = \frac{\partial f}{\partial \Omega_4} \Big|_h = -2B\Omega_4 = 2B\Omega_h; \quad \dot{Q} = \frac{\partial f}{\partial \Omega_1} \Big|_h = -A;$$

$$\dot{R} = \frac{\partial f}{\partial \Omega_2} \Big|_h = -A; \quad \dot{S} = \frac{\partial f}{\partial \Omega_3} \Big|_h = A; \quad \dot{R} = \frac{\partial f}{\partial \Omega_2} \Big|_h = -A;$$

Therefore,

$$\frac{d^2\psi}{dt^2} = \dot{L}\Delta\psi + \dot{M}\Delta\Omega_1 + \dot{N}\Delta\Omega_2 + \dot{O}\Delta\Omega_3 + \dot{P}\Delta\Omega_4 + \dot{Q}\Delta\frac{d\Omega_1}{dt} + \dot{R}\Delta\frac{d\Omega_2}{dt} + \dot{S}\Delta\frac{d\Omega_3}{dt} + \dot{T}\Delta\frac{d\Omega_4}{dt}$$

$$\frac{d^2\psi}{dt^2} = 2B\Omega_h(\Delta\Omega_1 - \Delta\Omega_2 + \Delta\Omega_3 - \Delta\Omega_4) + A(\Delta\frac{d\Omega_1}{dt} - \Delta\frac{d\Omega_2}{dt} + \Delta\frac{d\Omega_3}{dt} - \Delta\frac{d\Omega_4}{dt})$$

$$\frac{d^2\psi}{dt^2} = 0.004615(\Delta\frac{d\Omega_1}{dt} - \Delta\frac{d\Omega_2}{dt} + \Delta\frac{d\Omega_3}{dt} - \Delta\frac{d\Omega_4}{dt}) + 0.02619(\Delta\Omega_1 - \Delta\Omega_2 + \Delta\Omega_3 - \Delta\Omega_4)$$

For vertical acceleration,

$$\frac{d^2z}{dt^2} = 0.12206(V_1^2 + V_2^2 + V_3^2 + V_4^2)(\cos\theta\cos\varphi) - g$$

At operating point, it is assumed that all the angles are small or very close to 0 and so their cosines are approximately 1. Also, the vertical acceleration and velocity are 0, since the altitude of the vehicle is not changing.

We allowed A = 0.12206

$$\frac{d^2z}{dt^2} = A(V_1^2 + V_2^2 + V_3^2 + V_4^2)(\cos\theta\cos\varphi) - g = f(V_1, V_2, V_3, V_4)$$

The coefficients are:

$$\dot{L} = \frac{\partial f}{\partial z} \Big|_h = 0; \quad \dot{M} = \frac{\partial f}{\partial V_1} \Big|_h = 2AV_1 = 2BV_h; \quad \dot{N} = \frac{\partial f}{\partial V_2} \Big|_h = 2AV_2 = 2AV_h;$$

$$\ddot{O} = \frac{\partial f}{\partial V_3} \Big|_h = 2AV_3 = 2AV_h; \quad \dot{P} = \frac{\partial f}{\partial V_4} \Big|_h = 2AV_4 = 2AV_h;$$

$$\frac{d^2 Z}{dt^2} = \dot{L}\Delta Z + \dot{M}\Delta V_1 + \dot{N}\Delta V_2 + \ddot{O}\Delta V_3 + \dot{P}\Delta V_4$$

$$\frac{d^2 Z}{dt^2} = 2AV_h(\Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4)$$

$$\frac{d^2 Z}{dt^2} = 1.0937(\Delta V_1 + \Delta V_2 + \Delta V_3 + \Delta V_4)$$

Linearized Quad-rotor Model

The Simulink model of the linearized quad-rotor system is based on the equations below. The inputs of this model are the voltage control input of the motors whereas the output signals are the roll, pitch and yaw angular acceleration and the vertical acceleration.

$$\frac{d(\Delta\Omega)}{dt} = 155.521\Delta V - 5.909\Delta\Omega$$

$$\frac{d(\Delta\varphi)}{dt} = 2052.6496(\Delta V_2 - \Delta V_4)$$

$$\frac{d(\Delta\theta)}{dt} = 2052.6496(\Delta V_3 - \Delta V_1)$$

$$\frac{d^2\psi}{dt^2} = 0.004615\left(\Delta\frac{d\Omega_1}{dt} - \Delta\frac{d\Omega_2}{dt} + -\Delta\frac{d\Omega_3}{dt} + \Delta\frac{d\Omega_4}{dt}\right) + 0.02619(\Delta\Omega_1 - \Delta\Omega_2 + \Delta\Omega_3 - \Delta\Omega_4)$$

$$\frac{d^2\psi}{dt^2} = 1.0937(\Delta\Omega_1 - \Delta\Omega_2 + \Delta\Omega_3 - \Delta\Omega_4)$$

Transfer Functions of Linear Model

When the equations of motion are linearized the transfer function of quad-rotor plant can be easily obtained by using the Laplace transform.

The Transfer functions of the system, which presents the ratio of the output (angular velocity) to the input (motor voltage).

$$\frac{d\Omega}{dx} = 155.521\Delta V - 5.909\Delta\Omega$$

Using Laplace transform gives:

$$s\Omega(s) + 5.909\Omega(s) = 155.521V(s)$$

$$\frac{\Omega(s)}{V(s)} = \frac{155.521}{(s+5.909)}$$

Similarly, the rest of the transfer functions of the linearized quadcopter model obtained

$$\frac{d^2\varphi}{dt^2} = 2052.6496(\Delta V_2 - \Delta V_4)$$

$$s^2\varphi(s) = 2052.6496(V_2(s) - V_4(s))$$

$$\frac{\theta(s)}{V_4(s) - V_2(s)} = \frac{2052.6496}{s^2}$$

The same for linearization of the equation-of-motion Pitch

$$\frac{\theta(s)}{V_3(s) - V_1(s)} = \frac{2052.6496}{s^2}$$

For the yaw angular acceleration,

$$\frac{d^2\psi}{dt^2} = 0.004615\left(\Delta\frac{d\Omega_1}{dt} - \Delta\frac{d\Omega_2}{dt} + \Delta\frac{d\Omega_3}{dt} - \Delta\frac{d\Omega_4}{dt}\right) + 0.02619(\Delta\Omega_1 - \Delta\Omega_2 + \Delta\Omega_3 - \Delta\Omega_4)$$

$$s^2\psi(s) = 0.004615[s\Omega_1(s) - s\Omega_2(s) + s\Omega_3(s) - s\Omega_4(s)] + 0.02619[\Omega_1(s) - \Omega_2(s) + \Omega_3(s) - \Omega_4(s)]$$

$$s^2\psi(s) = 0.004615s[\Omega_1(s) - \Omega_2(s) + \Omega_3(s) - \Omega_4(s)] + 0.02619[\Omega_1(s) - \Omega_2(s) + \Omega_3(s) - \Omega_4(s)]$$

$$s^2\psi(s) = (0.004615s + 0.02619)[\Omega_1(s) - \Omega_2(s) + \Omega_3(s) - \Omega_4(s)]$$

$$\frac{\psi(s)}{[\Omega_1(s) - \Omega_2(s) + \Omega_3(s) - \Omega_4(s)]} = \frac{(0.004615s + 0.02619)}{s^2}$$

Since

$$\Omega(s) = \frac{155.521}{(s+5.909)}V(s)$$

then

$$\psi(s) = \frac{(0.718s+4.073)}{s^2(s+5.909)}V_1(s) - \frac{(0.718s+4.073)}{s^2(s+5.909)}V_2(s) + \frac{(0.718s+4.073)}{s^2(s+5.909)}V_3(s) - \frac{(0.718s+4.073)}{s^2(s+5.909)}V_4(s)$$

$$\frac{d^2Z}{dt^2} = 1.0937(\Delta V_1(s) - \Delta V_2(s) + \Delta V_3(s) - \Delta V_4(s))$$

$$s^2Z(s) = 1.0937[V_1(s) - V_2(s) + V_3(s) - V_4(s)]$$

$$Z(s) = \frac{1.0937}{s^2}V_1(s) + \frac{1.0937}{s^2}V_2(s) + \frac{1.0937}{s^2}V_3(s) + \frac{1.0937}{s^2}V_4(s)$$

The transfer functions obtained was implemented in Simulink and used to formulate the test PID controller for this review paper as well as any type of controller for any quadcopter